Observables of a test-mass along an inclined orbit in a post-Newtonian approximated Kerr spacetime to leading-order-quadratic-in-spin

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The orbital motion is derived for a non-spinning test-mass in the relativistic, gravitational field of a rotationally deformed body not restricted to the equatorial plane or spherical orbit. The gravitational field of the central body is represented by the Kerr metric, expanded to second post-Newtonian order including the linear and quadratic spin terms. The orbital period, the periastron advance, and the precession of the orbital plane are derived with the aid of novel canonical variables and action-based methods.

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I. INTRODUCTION

The motion of spinless particles in the gravitational field of a rotating body is an interesting and involved problem if the motion is not restricted to the equatorial plane of the central body. Ever since the discovery of quasi-periodic oscillations (QPOs) and horizontal-branch oscillations in X-ray sources such as neutron stars, microquasars or black holes, theorists have tried to explain the presence of these oscillations by identifying their frequencies with those of nodal precession frequency of ‘blobs’ in the accreting disks or Keplerian frequencies or periastron advance frequencies of the accreting rings. Since the mechanism that leads to these oscillations are associated with processes in the accretion disk, earlier calculations have been restricted to either close-to-equatorial plane or spherical or nearly circular orbits. Markovic in [5] studies the effect of equations-of-state of neutron stars on various orbital frequencies restricted to either equatorial plane or close to equatorial plane. Merloni *et al* in [19] calculate the Lense-Thirring precession frequency of test-masses restricted to spherical orbits (constant r) and identifies them with the frequencies of QPOs from black hole sources. Sibgatullin in [4] calculates the nodal and periastron-rotation frequencies for nearly circular, non-equatorial orbits. Comparison of the observed and theoretical frequencies, once the relation between them confirmed, also act as a tool to test the general theory of relativity in strongly gravitating regions. Exact analytic solutions for arbitrarily oriented orbits are known in case of rotating black holes [1–3] but the Newtonian analog or the post-Newtonian (pN) expansion of the results, say periastron shift, is not straightforward, see [4, 5]. Exact expression for the periastron advance of a test-mass in the gravitational field of a non-rotating black hole are well known, e.g. see [6–9]. Slow-motion or weak-field approximations (pN approximations) of highly analytical or involved calculations allows one to have a meaningful interpretation of the results. In the references e.g. [6], [10], and [11] the motion of spinning binary systems were studied analytically and periastron advances were calculated to higher pN orders. The role played by the Carter constant [12] in various problems of motion related with rotating black-holes is a quite interesting subject too, see [13, 14], where Carter-like constants were introduced. This procedure will also be necessary for our calculations, for which we use Hill-inspired canonical variables [15–17] being better suited for the problem in hand than the often applied Boyer-Lindquist coordinates [18].

We, for the first time, without restricting to any special case, calculate the orbital period, the intrinsic periastron advance, and the precession of the orbital plane of a test-mass in an inclined, generic orbit in a Kerr spacetime field approximated to second pN order for the monopole interaction, next-to-leading pN order linear in the spin parameter a, and the more important leading pN order quadratic in a^2 between the spinless test-mass and its spinning massive companion. For a highly inclined orbit with high eccentricity one needs to carefully handle the influence of the Carter constant which is absent in the case of equatorial orbits. We approach this problem by the most transparent way possible, which is by starting from an invariant scalar action in canonical variables allowing for a very clear definition and understanding of the observables deriving from it. For our calculations we use SI units. Our perturbative method for making cal-

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culations feasible is the post-Newtonian approximation technique, which expands field equations and equations of motion (EOMs) in inverse powers of c, the speed of light. Hence, if a term carries the power $\frac{1}{c^n}$, with $n \in \mathbb{N}_0$ it is referred to as being of the $\frac{n}{2}$th-pN order or $\frac{n}{2}$pN in short where 0pN is the usual Newtonian order.

II. THE UNDERLYING ACTION PRINCIPLE

The motion of the test-mass is governed by the binding energy, E_b, which is conserved, and defines a Hamiltonian for generating the equations of motion. The total energy E of the system is given by

$$E = mc^2 + E_b \tag{2.1}$$

with m being the rest-mass of the test-body. We calculate the binding energy to $O(c^{-5})$,

$$\begin{aligned}
E_b =& \frac{p_r^2}{2m} + \frac{\mathcal{C}^2}{2r^2m} - \frac{GMm}{r} + \frac{1}{c^2}\left[G\left(-\frac{\mathcal{C}^2M}{2r^3m} - \frac{3Mp_r^2}{2rm}\right)\right. \\
&\left. - \frac{p_r^4}{8m^3} - \frac{\mathcal{C}^4}{8r^4m^3} - \frac{\mathcal{C}^2p_r^2}{4r^2m^3} - \frac{G^2M^2m}{2r^2}\right] + \frac{1}{c^3}\frac{2\bar{a}G^2L_zM^2}{r^3} \\
&+ \frac{1}{c^4}\left[G\left(\frac{\mathcal{C}^4M}{8r^5m^3} + \frac{3\mathcal{C}^2Mp_r^2}{4r^3m^3} + \frac{5Mp_r^4}{8rm^3}\right) - \frac{G^3M^3m}{2r^3}\right. \\
&+ G^2\left(-\frac{\mathcal{C}^2M^2}{4r^4m} + \frac{3M^2p_r^2}{4r^2m}\right) + \frac{\mathcal{C}^6}{16r^6m^5} + \frac{3\mathcal{C}^4p_r^2}{16r^4m^5} + \frac{p_r^6}{16m^5} \\
&\left. + \frac{3\mathcal{C}^2p_r^4}{16r^2m^5} + \bar{a}^2G^2\left(-\frac{L_z^2M^2}{2r^4m} + \frac{M^2p_r^2}{2r^2m}\right)\right] + \mathcal{O}\left(\frac{1}{c^6}\right)
\end{aligned} \tag{2.2}$$

with the dimensionless spin parameter $\bar{a}\frac{GM}{c^2} = a$ within in the range $\bar{a} \in [-1, 1]$, L_z is the z-component of m's angular momentum, $\mathcal{C}$ is the Carter-like constant, r and p_r being the radial coordinate and its canonical conjugate momentum respectively. The Carter-like constant, $\mathcal{C}$, is defined as

$$\mathcal{C}^2 := L^2 + \alpha^2\bar{a}^2\cos^2\theta \qquad \text{with} \tag{2.3}$$

$$\begin{aligned}
\alpha^2 &= m^2c^2\left(1 - \frac{E^2}{m^2c^4}\right)\frac{r_S^2}{4} \\
&= -\frac{2E_b}{mc^2}\left(\frac{GMm}{c}\right)^2 + \mathcal{O}(E_b^2)\,.
\end{aligned} \tag{2.4}$$

We only consider leading order terms in $\bar{a}^2$, therefore, we can drop the E_b^2-terms being of higher pN order or next-to-leading-order. Notice that α^2 is positive because E_b is negative for bound orbits. This Carter-like constant appears naturally as a separation constant when using the Hamilton-Jacobi approach to the pN-expanded Kerr metric in Boyer-Lindquist coordinates. We use a coordinate system that has canonical Hill-inspired variables $(p_r, L, L_z; r, u, \Omega)$, with u being the true anomaly, i.e., the angle between the position vector $\mathbf{r} = r\mathbf{n}$ and the direction of the ascending node $\mathbf{N} = (\mathbf{e}_z \times \mathbf{L})/|\mathbf{e}_z \times \mathbf{L}|$: $\cos u = \mathbf{n}\cdot\mathbf{N}$ and $\sin u = \mathbf{n}\cdot\mathbf{W}$ with $\mathbf{W} = \mathbf{L}\times\mathbf{N}/L$. Ω is the angle of the ascending node as measured from the x-axis of the non-rotating orthonormal basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$: $\cos\Omega = \mathbf{N}\cdot\mathbf{e}_x$ and $\sin\Omega = \mathbf{N}\cdot\mathbf{e}_y$, measuring the precession of the orbital plane about L_z which is tilted from the equatorial plane with the inclination angle i, the angle between the background z-axis and the orbital angular momentum vector $\mathbf{L}$, so $L_z = L\cos i$. The standard canonical Poisson bracket relations are

$$\{r, p_r\} = \{u, L\} = \{\Omega, L_z\} = 1 \tag{2.5}$$

with all other brackets being zero. In contrast to the classic Hill variables $(\dot{r}, G, H; r, \tilde{u}, \tilde{\Omega})$ with $\mathbf{G} = \mathbf{r}\times\mathbf{v}$ ($\mathbf{v}$ being the coordinate velocity vector), $H = G_z$ and $(\tilde{u}, \tilde{\Omega})$ being conjugate to $\mathbf{G}$ and H respectively, we work with the momentum and its derived quantities $p_r = \mathbf{p}\cdot\mathbf{r}/r$, and $L = |\mathbf{L}|$ with $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, i.e. our (u, Ω) differs from $(\tilde{u}, \tilde{\Omega})$. They are equal in the Newtonian case (per unit mass), see [20] for variable relations between these different frames. Notice that our orbital plane is defined as being orthogonal to $\mathbf{L}$ and not to $\mathbf{G}$, where $\mathbf{G}$ is often denoted by $\mathbf{L}_N$ apart from a mass parameter. The corresponding action integral about a complete revolution from periastron to periastron reads

$$\begin{aligned}
S &= S(E_b, L_z, \mathcal{C}, P, \Phi, U) \\
&= -E_bP + L_z\Phi + \oint p_r\,\mathrm{d}r + \oint L\,\mathrm{d}u\,.
\end{aligned} \tag{2.6}$$

It depends on the three constants of motion, $E_b, L_z, \mathcal{C}$, and three orbital-completed variables

$$P = \oint \mathrm{d}t\,, \quad \Phi = \oint \mathrm{d}\Omega \quad \text{and} \quad U = \oint \mathrm{d}u\,. \tag{2.7}$$

P being the orbital period, Φ the precession of the orbital plane per revolution, and U the intrinsic periastron advance. The relation between the Hill-inspired variable u and the Boyer-Lindquist coordinate θ [21] is

$$\sin u \sin i = \cos\theta\,. \tag{2.8}$$

Solving Eq(2.2) for p_r and Eq(2.3) for L allows the straightforward calculation of the action, Eq(2.6). Within the context of our leading order calculation, L in $\cos i$ can be replaced by $\mathcal{C}$. The next step is to calculate the remaining integrals in the action (2.6). We start with

$$\begin{aligned}
\oint L\mathrm{d}u &= \oint \sqrt{\mathcal{C}^2 - \alpha^2\bar{a}^2\sin^2 i\,\sin^2 u}\,\mathrm{d}u \\
&= \oint \mathcal{C}\mathrm{d}u - \oint \frac{1}{2\mathcal{C}}\bar{a}^2\alpha^2\sin^2 i\,\sin^2 u\,\mathrm{d}u + \mathcal{O}(\bar{a}^4) \\
&= \mathcal{C}U - \frac{\pi}{2\mathcal{C}}\bar{a}^2\alpha^2\left(1 - \frac{L_z^2}{\mathcal{C}^2}\right) + \text{NLO-}\bar{a}^2\text{-terms}\,.
\end{aligned} \tag{2.9}$$

Since the integrand is of leading-order in $\bar{a}^2$ we integrate over a closed Newtonian orbit, i.e., we integrate from 0 to

2π and drop all next-to-leading-order (NLO) corrections. The action, adapted to our problem, reads

$$S = -E_b P + L_z \Phi + \oint p_r \mathrm{d}r + \mathcal{C} U - \frac{\pi}{2\mathcal{C}} \bar{a}^2 \alpha^2 \left(1 - \frac{L_z^2}{\mathcal{C}^2}\right) . \tag{2.10}$$

The action principle tells us

$$\frac{\partial S}{\partial \mathcal{C}} = \frac{\partial S}{\partial L_z} = \frac{\partial S}{\partial E_b} = 0 \tag{2.11}$$

giving rise to formulas for the orbital elements,

$$U = -\frac{\partial}{\partial \mathcal{C}} \oint p_r \mathrm{d}r - \frac{\bar{a}^2 \pi \alpha^2 (E_b)}{2\mathcal{C}^2} \left(1 - \frac{3L_z^2}{\mathcal{C}^2}\right) , \tag{2.12}$$

$$\Phi = -\frac{\partial}{\partial L_z} \oint p_r \mathrm{d}r - \bar{a}^2 \pi \alpha^2 (E_b) \frac{L_z}{\mathcal{C}^3} , \tag{2.13}$$

$$P = \frac{\partial}{\partial E_b} \oint p_r \mathrm{d}r + \frac{\bar{a}^2 \pi G^2 M^2 m}{c^4 \mathcal{C}} \left(1 - \frac{L_z^2}{\mathcal{C}^2}\right) . \tag{2.14}$$

The integral over p_r can be evaluated perturbatively in inverse powers of c, where p_r is evaluated by inverting Eq(2.2) to $O(c^{-5})$. To make the integration easy we choose the method of contour integration employed by Sommerfeld in [22],

$$\oint p_r \mathrm{d}r = -2\pi i \left[\mathrm{Res}_{r=0}(p_r) + \mathrm{Res}_{r=\infty}(p_r)\right] \tag{2.15}$$

The result reads, also see [6, 23]:

$$\begin{aligned}
\frac{1}{2\pi} \oint p_r \mathrm{d}r &= -\mathcal{C} + \frac{GMm}{\sqrt{-\frac{2E_b}{m}}} \\
&+ \frac{1}{c^2} \left(\frac{3G^2M^2m^2}{\mathcal{C}} + \frac{15GMm}{4} \sqrt{\frac{-E_b}{2m}} \right) \\
&- \frac{1}{c^3} \frac{2\bar{a} G^3 L_z M^3 m^3}{\mathcal{C}^3} \\
&+ \frac{1}{c^4} \left[\frac{15 E_b G^2 M^2 m^4}{2\mathcal{C}} + \frac{35 G^4 M^4 m^4}{4\mathcal{C}^3} + \frac{35 E_b GM}{32} \sqrt{-\frac{E_b}{2m}} \right. \\
&+ \bar{a}^2 \left(E_b G^2 \left(-\frac{M^2 m}{2\mathcal{C}} + \frac{L_z^2 M^2 m}{2\mathcal{C}^3} \right) \right. \\
&\left. \left. - \frac{G^4 M^4 m^4}{4\mathcal{C}^3} + \frac{3 G^4 L_z^2 M^2 m}{2\mathcal{C}^5} \right) \right] \\
&- \frac{\bar{a}}{c^5} \left(\frac{12 E_b G^3 M^3 L_z m^2}{\mathcal{C}^3} + \frac{21 G^5 M^5 L_z m^5}{\mathcal{C}^5} \right) .
\end{aligned} \tag{2.16}$$

The two periastron shifts, $\Delta\bar{U}$ and $\Delta\bar{\Phi}$, the intrinsic and the one related with the precession of the orbital plane, respectively, are then giving by

$$\begin{aligned}
\Delta\bar{U} &= \frac{1}{2\pi}(U - 2\pi) \\
&= \frac{3G^2M^2m^2}{c^2\mathcal{C}^2} - \frac{6\bar{a}G^3M^3m^3}{c^3\mathcal{C}^3} \cos i \\
&+ \frac{1}{c^4} \left[\frac{15 E_b G^2 M^2 m}{2\mathcal{C}^2} + \frac{105 G^4 M^4 m^4}{4\mathcal{C}^4} \right. \\
&\left. + \bar{a}^2 \frac{3G^4M^4m^4}{4\mathcal{C}^4} \left(5\cos^2 i - 1\right) \right] \\
&- \frac{\bar{a} G^3 M^3 m^2}{c^5 \mathcal{C}^3} \left(36 E_b + 105 \frac{G^2 M^2 m^3}{\mathcal{C}^2} \right) \cos i
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
\Delta\bar{\Phi} = \frac{1}{2\pi}\Phi &= \frac{2\bar{a}G^3M^3m^3}{c^3\mathcal{C}^3} + \frac{\bar{a}^2}{c^4} \cos i \left(-\frac{3G^4M^4m^4}{2\mathcal{C}^4} \right) \\
&+ \frac{\bar{a}}{c^5} \left(\frac{12 E_b G^3 M^3 m^2}{\mathcal{C}^3} + \frac{21 G^5 M^5 m^5}{\mathcal{C}^5} \right) .
\end{aligned} \tag{2.18}$$

We also recover the usual pN result when the motion takes place in the equatorial plane. In this case, $\mathcal{C} = L = L_z$ and our precession angle Φ agrees to leading pN order with Eq. (43) in [5], where it is called the nodal precession rate. Furthermore, the leading-order linear-in-$\bar{a}$ terms of $\Delta\bar{U}$ and $\Delta\bar{\Phi}$ fully agree with the results in [6] for the case of an inclined orbital plane when referred to the test-mass limit. For the period we have

$$\begin{aligned}
P = \frac{2\pi GM}{(-2E_b/m)^{3/2}} &\left[1 - \frac{1}{c^2} \frac{15}{4} \frac{E_b}{m} \right. \\
&+ \frac{1}{c^4} \left(\frac{15}{2} \frac{GMm}{\mathcal{C}} \left(\frac{-2E_b}{m} \right)^{3/2} - \frac{105}{32} \frac{E_b^2}{m^2} \right) \\
&\left. - \frac{12\bar{a}G^2M^2m^2}{c^5\mathcal{C}^2} \left(\frac{-2E_b}{m} \right)^{3/2} \cos i \right] .
\end{aligned} \tag{2.19}$$

In the case of equatorial motion the resulting shift, $\Delta\tilde{\Phi}$, (for this degenerate case) is given by

$$\begin{aligned}
\Delta\tilde{\Phi} &= \Delta\bar{U} + \Delta\bar{\Phi} = -1 - \frac{1}{2\pi} \left(\frac{\partial}{\partial \mathcal{C}} + \frac{\partial}{\partial L_z} \right) \oint p_r \mathrm{d}r \\
&= \frac{3G^2M^2m^2}{c^2\mathcal{C}^2} - \frac{4\bar{a}G^3M^3m^3}{c^3\mathcal{C}^3} \\
&+ \frac{1}{c^4} \left[\frac{15 E_b G^2 M^2 m}{2\mathcal{C}^2} + \frac{105 G^4 M^4 m^4}{4\mathcal{C}^4} + \bar{a}^2 \frac{3G^4M^4m^4}{2\mathcal{C}^4} \right] \\
&+ \frac{\bar{a}}{c^5} \left(-\frac{24 E_b G^3 M^3 m^2}{\mathcal{C}^3} - \frac{84 G^5 M^5 m^5}{\mathcal{C}^5} \right) .
\end{aligned} \tag{2.20}$$

This quantity has been calculated by other methods [11] whose result coincides with ours, also see [5] to leading-pN-order. Notice that one can replace the Carter-like constant $\mathcal{C}$ by L in the Eqs. (2.17)-(2.20) because the $\bar{a}^2$-correction in $\mathcal{C}$ is lifting the pN order of the corresponding terms beyond our considerations.

III. APPLICATION TO MEASURED DATA

As an application, we calculate the orbital elements of star S2 (treated as test-mass) orbiting the radio source Sagittarius A*, an assumed supermassive black hole with the mass $M = 4 \cdot 10^6 M_\odot$ [24] and a spin parameter $\bar{a} = 0.44$ [25]. We take the data of S2 star from SIMBAD. The data are

Data label	Measured value
Gravitat. constant	$G = 6.67384(80) \cdot 10^{-11}\ \mathrm{m}^3/(\mathrm{kg}\cdot\mathrm{s}^2)$
Solar mass	$M_\odot = 1.98855(25) \cdot 10^{30}$ kg
Semimajor axis	980 AU
Astronomical Unit	$149.60 \cdot 10^9$ km
Eccentricity	0.881
Inclination angle	$i = -48.1°$
Speed of light	$c = 299792458$ m/s

The analysis of the orbital elements is as follows. We first give the Newtonian value labelled with N, then the pN monopole corrections to it with the label 0, followed by the spin-orbit corrections (leading and, formally, next-to-leading order) labelled by 1 and finally the spin-squared corrections labelled by 2.

P (S2)	$\Delta\bar{U}$ (S2)	$\Delta\bar{\Omega}$ (S2)
$P_\mathrm{N} = 15.3484$ yr	$\Delta\bar{U}_\mathrm{N} = 0$	$\Delta\bar{\Omega}_\mathrm{N} = 0$
$P_0 = 0.0012$ yr	$\Delta\bar{U}_0 = 701.115''$	$\Delta\bar{\Omega}_0 = 0$
$P_1 = -2.5 \cdot 10^{-9}$ yr	$\Delta\bar{U}_1 = -5.537''$	$\Delta\bar{\Omega}_1 = 2.761''$
$P_2 = 0$ yr	$\Delta\bar{U}_2 = 0.008''$	$\Delta\bar{\Omega}_2 = -0.008''$

As one sees, the spin effects are negligible and within the error bar for the period $P = (15.56 \pm 0.35)$yr. Though we have used Hill-inspired variables to calculate the orbital elements only to some pN order, it will be interesting to see if these canonical variables would be advantageous to calculate the orbital elements non-perturbatively starting from a non-pN-approximated Kerr spacetime field as was done in Boyer-Lindquist coordinates by [26, 27]. At least in the case of equatorial motion, the orbital elements calculated in Hill-inspired variables should match the orbital frequencies given therein.

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